Blow-Up of Solutions for a Quasilinear Parabolic Equation with Logarithmic Nonlinearity Term

Abstract: This paper deals with a quasilinear parabolic equation with p-Laplacian and logarithmic nonlinearity terms under homogeneous Dirichlet boundary condition in a smooth bounded domain. By means of the logarithmic Sobolev inequality and potential wells method, the authors obtain a blow-up result.

Keywords: Quasilinear parabolic equation; Logarithmic nonlinearity; Logarithmic Sobolev inequality; Potential wells method; Blow-up.

INTRODUCTION

In this paper we consider the following quasilinear parabolic equation with p-Laplacian and logarithmic nonlinearity terms

$$\begin{cases}
\Delta_t u - \nabla u^p = |u|^{p-2}u \log |u|, & (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0, & (x,t) \in \partial \Omega \times (0,T), \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
$$

where $\Omega \subset R^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$, $p > 2$, $u_0(x) \in \mathcal{W}^{1,p}_{0}(\Omega) \setminus \{0\}$.


In recent years, parabolic and hyperbolic type equations with logarithmic nonlinearity or logistic source have been studied by many authors (see [7-20] and the references therein). In the case $p = 2$, problem (1) has been studied by in (Chen, H., Luo, P., & Liu, G. 2015; & Han, Y. 2019). In (Chen, H., Luo, P., & Liu, G. 2015), Chen et al., investigated the following semilinear heat equation with logarithmic nonlinearity

$$u_t - \Delta u = u \log |u|,$$

in a bounded domain $\Omega \subset R^n (n \geq 1)$ with homogeneous Dirichlet boundary condition.

By using the logarithmic Sobolev inequality and a family of potential wells, the authors obtained the existence of global solution and blow-up at $+\infty$ under some suitable conditions. On the other hand, the results for decay estimates of the global solutions are also given. The author in (Han, Y. 2019) improved a blow-up result obtained in (Chen, H., Luo, P., & Liu, G. 2015).

In (Truong, L. X. 2017), the authors considered the following model of a nonlinear pseudoparabolic equation with logarithmic nonlinearity:

$$\begin{cases}
\Delta_t u - \Delta u - \Delta_p u = |u|^{p-2}u \log |u|, & x \in \Omega, \ t > 0, \\
u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
$$

in which $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$. By using the potential well method and a logarithmic Sobolev inequality, the authors obtained results of existence or nonexistence of global weak solutions. In addition, the authors...
also provide sufficient conditions for the large time decay of global weak solutions and the finite time blow-up of weak solutions.

The authors in (Toualbia, S. et al., 2020) considered the Neumann problem to the following initial parabolic equation with logarithmic source:

$$
\begin{aligned}
&u_t - \text{div}(\nabla u |\nabla u|^{p-2} \nabla u) = |u|^{p-2} u \log |u| - \int_\Omega |u|^{p-2} u \log |u| dx, \quad x \in \Omega, \; t > 0, \\
&\frac{\partial u}{\partial n}(x, t) = 0, \quad x \in \partial \Omega, \; t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
$$

(3)

in a bounded domain with smooth boundary, $p > 2$. By using the logarithmic Sobolev inequality and potential wells method, they obtain the decay, blow-up and non-extinction of solutions under some conditions.

In (Ding, H., & Zhou, J. 2019), the following model of a quasilinear diffusion equation with interior logarithmic source has been studied:

$$
\begin{aligned}
&u_t - \text{div}(\nabla u |\nabla u|^{p-2} \nabla u) = |u|^{p-2} u \log |u|, \quad x \in \Omega, \; t > 0, \\
&u(x, t) = 0, \quad x \in \partial \Omega, \; t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
$$

(4)

in which $p > 2$, $u_0(x) \in W_0^{1,p}(\Omega) \setminus \{0\}$. By using the potential well method and a logarithmic Sobolev inequality, the authors obtained results of existence or nonexistence of global weak solution. They also provided sufficient conditions for the large time decay of global weak solutions and for the finite time blow-up of weak solutions. Among some other interesting results, they showed that the weak solution $u(x, t)$ of problem (4) blows up at finite time under the condition $J(u_0) \leq M$ and $I(u_0) < 0$, here $M > 0$ is a constant, the energy functional $J(u)$ and Nehari functional $I(u)$ are defined as follows

$$
J(u) = \frac{1}{p} ||\nabla u||_p^p - \frac{1}{p} \int_\Omega |u|^{p-2} u \log |u| dx + \frac{1}{p^2} ||u||_p^p,
$$

(5)

$$
I(u) = ||\nabla u||_p^p - \int_\Omega |u|^{p-2} u \log |u| dx,
$$

(6)

in which $|| \cdot ||_p = (\int_\Omega | \cdot |^p dx)^{\frac{1}{p}}$. Moreover, we have

$$
J(u) = -\frac{1}{p} I(u) + \frac{1}{p^2} ||u||_p^p.
$$

(7)

Motivated by the above studies, in this paper we investigate blow up of solutions of problem (1). We will give new conditions of $J(u_0)$ for the weak solution of problem (1) to blow up which extends the results in recent literatures (Han, Y. 2019; Toualbia, S. Toualbia, S. et al., 2020; & Ding, H., & Zhou, J. 2019).

2. MAIN RESULT

To establish main results requires following lemmas and definition. First, we show the logarithmic Sobolev inequality which is a fundamental tool for the following estimates.

**Lemma 2.1.** (Zhou, J. 2019)

Let $p > 1$, $\mu > 0$ and $u(x) \in W_0^{1,p}(\mathbb{R}^n) \setminus \{0\}$. Then

$$
p \int_{\mathbb{R}^n} |u(x)|^p \log\left(\frac{|u(x)|}{||u||_{L_p(\mathbb{R}^n)}}\right) dx + \frac{n}{p} \log\left(\frac{p \mu e}{n \ell_p}\right) \int_{\mathbb{R}^n} |u(x)|^p dx \leq \mu \int_{\mathbb{R}^n} |\nabla u(x)|^p dx,
$$

where

$$
\ell_p = \frac{p}{n} (\frac{p-1}{e})^{p-1} p^{-\frac{n}{p}} \left(\frac{\Gamma(\frac{n}{p} + 1)}{\Gamma(n \frac{p-1}{p} + 1)}\right)^\frac{1}{n}.
$$

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Remark

If \( u(x) \in W_0^{-1, p}(\Omega) \), by defining \( u(x) = 0 \) for \( x \in \mathbb{R}^n \setminus \Omega \), then
\[
p \int_{\Omega} |u(x)|^p \log \left( \frac{|u(x)|}{||u||_p} \right) dx + \frac{n}{p} \log \left( \frac{\mu e}{n \ell_p} \right) \int_{\Omega} |u(x)|^p dx \leq \mu \int_{\Omega} |\nabla u(x)|^p dx, \tag{8}
\]
for any real number \( \mu > 0 \).

Lemma 2.2. (Zhou, J. 2019)

Let \( u_0 \in X_0 \). Then there exists a positive constant \( T_c \) such that problem (1) has a weak solution \( u(x, t) \) on \( \Omega \times [0, T_0) \). Furthermore, \( u(x, t) \) satisfies the energy inequality
\[
\int_{0}^{t} ||u_\sigma(\cdot, s)||_2^2 ds + J(u(x, t)) \leq J(u_0), \quad t \in [0, T_0). \tag{9}
\]

Lemma 2.3. ([10])

Suppose that \( \theta > 0 \), \( \alpha > 0 \), \( \beta > 0 \) and \( h(t) \) is a nonnegative and absolutely continuous function satisfying
\[
h'(t) + \alpha h^\theta(t) \geq \beta, \quad \text{then for} \quad 0 < t < +\infty, \quad \text{it holds}
\]
\[
h(t) \geq \min \{h(0), \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\theta}} \}.
\]

Definition 2.1. (Weak solution) (Zhou, J. 2019)

A function \( u = u(x, t) \in L^\infty(0, T; X_0) \) with \( u_t \in L^{p'}(0, T; W^{-1, p'}(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) is called a weak solution to problem (1) in \( \Omega \times [0, T) \), if \( u(x, 0) = u_0(x) \in X_0 \) and \( u(x, t) \) satisfies (1) in the sense of distribution, i.e.
\[
\int_{\Omega} u_t \omega dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \omega dx = \int_{\Omega} |u|^{p-2} u \log |u| \omega dx, \tag{10}
\]
for all \( \omega \in W_0^{-1, p}(\Omega), \; t \in (0, T) \), where \( X_0 = W_0^{1, p}(\Omega) \setminus \{0\} \), \( W^{-1, p'}(\Omega) \) to denote the dual space of \( W_0^{1, p}(\Omega) \), \( p' \) is Holder conjugate exponent of \( p > 1 \).

Definition 2.2. (Blow up at infinity)

Let \( u(x, t) \) be a weak solution of (1), we call \( u(x, t) \) blow up at \( +\infty \) if the maximal existence time \( T' = +\infty \) and
\[
\lim_{t \to +\infty} ||u(\cdot, t)||_2 = \infty. \tag{11}
\]

Our main results are the following,

Theorem 2.1. Assume that \( u_0 \in X_0 \) and \( J(u_0) < 0 \). Then the weak solution \( u = u(x, t) \) of problem (1) blows up at infinity. Moreover, if \( ||u(t)||_2 \leq \left( \frac{-pJ(u_0)}{pJ'(u_0)} \right)^{\frac{1}{p}} \), the lower bound for blow up rate can be estimated by
\[
||u||_2^2 \geq ||u_0||_2^2. \tag{12}
\]

Proof. Assume that \( u(x, t) \) is the weak solution of problem (1). Set \( G(t) = ||u(\cdot, s)||_2^2 \), then
\[
G'(t) = 2 \int_{\Omega} u_t u dx = -2I(u(x, t)) = -2pJ(u) + \frac{2}{p} ||u||_p^p \geq -2pJ(u), \tag{13}
\]
By (9) in Lemma 2.2 and the condition \( J(u_0) < 0 \), we can get
\[
-2pJ(u) \geq 2p \int_{0}^{t} ||u_\sigma(\cdot, s)||_2^2 ds. \tag{14}
\]
Then by (13) and (14), we have
\[
G'(t) \geq 2p \int_{0}^{t} ||u_\sigma(\cdot, s)||_2^2 ds. \tag{15}
\]
By the Definition 2.1, for any $t_0 > 0$, we can prove
\[
\int_{t_0}^\infty \|u_s(\cdot, s)\|^2_2 ds > 0.
\]
(16)

If the above formula does not hold, i.e., there exists $t_0 > 0$ such that $\int_{t_0}^\infty \|u_s(\cdot, s)\|^2_2 ds = 0$. This implies that $u_t = 0$ for a.e. $(x, t) \in \Omega \times (0, t_0)$.

From (15), we can get $-|\nabla u|^p + |u|^p \log |u| = 0$, for a.e. $t \in (0, t_0)$. By the definition of $J(u)$ and (9), we have
\[
J(u) = \frac{1}{p^2} \|u\|^p_p \leq J(u_0) \leq 0,
\]
which implies that $\|u\|^p_p = 0$ for all $t \in (0, t_0)$; contradicts the definition of $u$, then (16) is true.

Fix $t_0 > 0$ and let $\kappa = \int_{t_0}^t \|u\|^2_2 ds$, this implies that $\kappa$ is a positive constant.

Integrating (15) over $(t_0, t)$, we can get
\[
G(t) \leq G(t_0) + 2p \int_{t_0}^t \|u_s(\cdot, s)\|^2_2 ds \geq G(t_0) + 2p \int_{t_0}^t \|u_t\|^2_2 ds \geq 2p \kappa (t - t_0).
\]
Hence
\[
\lim_{t \to +\infty} G(t) = \infty.
\]
(18)

This means that the weak solution $u = u(x, t)$ of problem (1) blows up at infinity.

From (9) and (13), we have
\[
G'(t) = -2pJ(u) + \frac{2}{p} \|u\|^p_p - 2pJ(u_0) + \frac{2}{p} \|u\|^p_p,
\]
(19)
\[
G'(t) + lG^\frac{p}{2}(t) \geq -2pJ(u_0) + \frac{2}{p} \|u\|^p_p + lG^\frac{p}{2}(t) \geq -2pJ(u_0) + \frac{2}{p} \|u\|^p_p + l\|u\|^p_2
\]
\[
\geq \frac{l(p + 2)}{p} \|u\|^p_p - pJ(u_0) \geq -pJ(u_0),
\]
(20)
in which $J$ is a constant in the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega), \ p > 2$.

By Lemma 2.3, $J(u_0) < 0$ and $\|u_0\|^2_2 \leq \left( \frac{-pJ(u_0)}{l} \right)^\frac{2}{p}$, we have
\[
G(t) \geq \min \{\|u_0\|^2_2, \left( \frac{-pJ(u_0)}{l} \right)^\frac{2}{p} \} \geq \|u_0\|^2_2,
\]
this ends the proof.

**Theorem 2.2.** Assume that $u_0 \in X_0, J(u_0) < 0$ and $J(u) < 0$. Then the weak solution $u = u(x, t)$ of problem (1) blows up at finite time.

**Proof.**

Assume that $u(x, t)$ is the weak solution of problem (1). we prove that the solution $u(x, t)$ is not global, that means, it blows up at finite time. Assume by contradiction that the solution $u(x, t)$ is global.

By the definition of $I(u)$, we have
\[
I(u) = (1 - \frac{\mu}{p})\|\nabla u\|^p_p + \frac{\mu}{p} \|\nabla u\|^p_p - \int_\Omega |u|^p \log \frac{|u|}{\|u\|^p_p} dx = \int_\Omega |u|^p \log \|u\|^p_p dx.
\]
Choosing $\mu = p$, and we apply the logarithmic Sobolev inequality (Lemma 2.1), we obtain
If $\tilde{I}(u) < 0$, we can get

$$||u||_p > \left( \frac{p^2 e}{\ln^p} \right)^{\frac{1}{p^2}} := R. \quad (21)$$

Then, for any $T > 0$, we define the functional

$$\Gamma(t) = \int_0^t ||u(\cdot, s)||_2^2 ds + (T - t)||u_0||_2^2, \quad t \in [0, T]. \quad (22)$$

It is obviously that $\Gamma(t) > 0$ for all $t \in [0, T]$. Hence, since $\Gamma$ is continuous, there exists $\rho > 0$ (independent of $T$) such that $\Gamma(t) > \rho$ for all $t \in [0, T]$. Then we have

$$\Gamma'(t) = ||u(\cdot, t)||_2^2 - ||u_0||_2^2, \quad (23)$$

and

$$\Gamma''(t) = 2\int_\Omega u_t u d\sigma = -2I(u(x, t)) = -2pJ(u) + \frac{2}{p}||u||_p^p. \quad (24)$$

By using (9) in Lemma 2.2, we have

$$-2pJ(u) \geq 2p\int_0^t ||u(\cdot, s)||_2^2 ds - 2pJ(u_0). \quad (25)$$

From $J(u_0) < 0$, (21), (24), (25), we get

$$\Gamma''(t) \geq 2p\int_0^t ||u(\cdot, s)||_2^2 ds - 2pJ(u_0) + \frac{2}{p}||u||_p^p \geq 2p\left( \frac{R^p}{p^2} - J(u_0) \right) + \int_0^t ||u(\cdot, s)||_2^2 ds \geq 2p\left( \frac{R^p}{p^2} \right) + \int_0^t ||u(\cdot, s)||_2^2 ds. \quad (26)$$

Now, multiplying (26) by $\Gamma(t)$, we have

$$\Gamma''(t)\Gamma(t) \geq 2p\left( \frac{R^p}{p^2} \right) + \int_0^t ||u(\cdot, s)||_2^2 ds \Gamma(t) = 2p\left( \frac{R^p}{p} \right) \Gamma(t) + 2p\int_0^t ||u(\cdot, s)||_2^2 ds \int_0^t ||u(\cdot, s)||_2^2 ds. \quad (27)$$

Noticing that

$$\Gamma'(t) = ||u(\cdot, t)||_2^2 - ||u_0||_2^2 = \int_0^t \frac{d}{ds} \left( ||u(\cdot, s)||_2^2 \right) ds = 2\int_0^t \int_\Omega u_s u d\sigma ds,$$

then

$$\left( \Gamma'(t) \right)^2 = 4\left( \int_0^t \int_\Omega u_s u d\sigma ds \right)^2.$$

With the help of Cauchy-Schwarz inequality, we have

$$\left( \Gamma'(t) \right)^2 \leq 4\int_0^t ||u_s(\cdot, s)||_2^2 ds \int_0^t ||u(\cdot, s)||_2^2 ds. \quad (28)$$

By using (27) and (28), we further obtain

$$\Gamma''(t)\Gamma(t) - \frac{p}{2} \left( \Gamma'(t) \right)^2 \geq 2\left( \frac{R^p}{p} \right) \rho > 0. \quad (29)$$

By setting $y(t) = \Gamma(t)^{-(p-2)/2}$, the above inequality (29) becomes
\[ y''(t) \leq -\delta y(t) \]

in which \( \delta = \frac{(p-2)pR_0}{d} \). This proves that \( y(t) \) reaches 0 in finite time, say as \( t \to T^* \).

And therefore,

\[ \lim_{t \to T^*} \Gamma(t) = \infty. \]

Which implies

\[ \lim_{t \to T^*} \int_0^t \| u(\cdot, s) \|_2^2 ds = \infty. \]

As a consequence, we get

\[ \lim_{t \to T^*} \| u(\cdot, t) \|_2^2 ds = \infty, \]

this means \( u(\cdot, t) \) blows up at finite time \( T^* \).

3. Conclusion

In this work, by using the logarithmic Sobolev inequality and potential wells method, we study the initial boundary value problem of a quasilinear parabolic equation with logarithmic nonlinearity in a bounded domain, where we obtain the blow-up of solutions under some new conditions.

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